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Abstract

As a stochastic model for quantum mechanics we present a stationary quantum Markov process for the time evolution of the Wigner function on a lattice phase space $Z_N \times Z_N$ with N odd. By introducing a phase factor extension to the phase space, each particle can be treated independently. This is an improvement on earlier methods that require the whole distribution function to determine the evolution of a constituent particle. The process has branching and vanishing points, though a finite time interval can be maintained between the branchings. The procedure to perform a simulation using the process is presented.

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1. Introduction

Various studies of quantum mechanics have been made by using trajectories. For example, in Feynman's path integral method, the transition amplitude between the initial and final points is calculated by adding all the phase factors determined from the trajectories connecting these points [1]. In this case, the trajectories are interfering alternatives and not those of observed particles. The method proposed by Bohm is another example. In this method, the trajectories are treated as if they were observed and are calculated deterministically [2]. These methods give alternative formulations of quantum mechanics.

The stochastic approach is another method. It is used not only for the formulation of quantum mechanics, but also for practical calculations. The use of a stochastic method for practical calculations is generally called the quantum Monte Carlo method (QMC) and is widely used in the simulation of quantum systems. Typical algorithms in the QMC are the random walk Monte Carlo method (RWMC), the diffusion Monte Carlo method (DMC) and the path integral Monte Carlo method (PIMC). These methods are efficient ways of determining the wavefunction and energy of a ground state [3, 4]. They are based on the formal analogy

that the Schrödinger equation becomes the diffusion equation when t is substituted by an imaginary time $i\tau$.

A solution of the diffusion equation can be constructed using Brownian motion. We can generate sample paths of Brownian particles by a random number generator and can follow the paths independently as Brownian motion is a stationary Markov process, which means that the diffusion equation can be solved by a stochastic method using a random number generator. Many studies have been conducted on the diffusion equation and Brownian motion. It is known that a stochastic estimation is generally more efficient than direct calculations when the system has many degrees of freedom. In direct methods, the amount of calculation required generally increases exponentially with the number of degrees of freedom D [5]. On the other hand, for stochastic methods the amount of calculation is of order D for one trial, though the estimated error is of order $1/\sqrt{n}$ where n is the number of trials. Very naively, the Schrödinger equation is a diffusion equation along an imaginary time axis. However, the quantum stationary Markov process for the Schrödinger equation, which corresponds to Brownian motion for the diffusion equation, has not yet been established. Formulating an analogous process to Brownian motion in quantum mechanics would be useful for estimating the time development of wavefunctions, especially in higher degrees of freedom.

Because of the probabilistic nature of quantum mechanics, several stochastic formulations of quantum mechanics have been investigated [6, 7]. A typical example of such a stochastic process was formulated by Nelson in 1966 [6]. In this method, the behavior of the constituent particles is determined by a stochastic differential equation and their distribution function coincides with the absolute square of the wavefunction. In this sense, the trajectories in this method are those of the observed particles. We can apply this method to estimate the tunneling time by making simulations of the trajectories [8]. However, the drift term in the stochastic equation is calculated from the wavefunction, and hence the Schrödinger equation must be solved to construct the process. A general method to construct a stochastic process without solving the Schrödinger equation was developed by Guerra and Marra [9]. In their method, the total distribution of particles is required to determine the next step of a constituent particle, though the obtained process is a Markov process. It is possible to determine the time evolution of all particles simultaneously, but each individual particle cannot be treated independently. These methods treat the distribution function directly, i.e. the absolute square of a wavefunction.

A way to avoid this difficulty is to search for a stochastic process for the time evolution equations which are equivalent to the Schrödinger equation. One such equation is the quantum Liouville equation for the Wigner function. In this paper, we discuss a stationary quantum stochastic process for the Wigner function in a lattice phase space. A similar approach was considered by Cohendet *et al* [10]. They constructed the Wigner function on a lattice constituting of an odd number of lattice points, and found a background quantum Markov process.

The Wigner function is defined by Wigner as

$$\rho(q, p) = \frac{1}{\pi\hbar} \int_{-\infty}^{\infty} dr\, e^{2ipr/\hbar} \psi^*(q + r)\psi(q - r) \tag{1}$$

in the phase space, where $\psi(q)$ is a wavefunction [11, 12]. The Wigner function gives the density distribution in the configuration space if it is integrated along the momentum direction:

$$\int_{-\infty}^{+\infty} \rho(q, p)\, dp = |\psi(q)|^2, \tag{2}$$

and gives the density distribution in the momentum space if integrated along the configuration direction:

$$\int_{-\infty}^{+\infty} \rho(q, p)\, \mathrm{d}q = |\tilde{\psi}(p)|^2. \tag{3}$$

This property is the so-called marginality of the Wigner function. In this sense, the Wigner function can be regarded as a kind of *distribution function* in the phase space and plays an important role in quantum mechanics, especially in the quantum-classical correspondence. However, the value of the Wigner function can be negative, so that it is not a distribution function itself.

Cohendet *et al* regarded a lattice with odd N lattice points as a cyclic group Z_N of order N and constructed a Wigner function in the lattice phase space $Z_N \times Z_N$, i.e. the direct product of Z_N and its dual ($\cong Z_N$). The Wigner function can also be negative on a finite lattice, but the value is bounded. They extended the phase space $Z_N \times Z_N$ by a dichotomic variable σ of value ± 1 and considered a new function on the extended space adding a constant to the Wigner function. The function is normalized and positive as the original function is bounded, so it can be regarded as a distribution function of particles in the new space. They derived the time evolution equation for the distribution function from the quantum Liouville equation, which determines the time evolution of the Wigner function, and constructed the background stochastic process of the particles using the Guerra–Marra method. Although the obtained process has a Markov property, the whole distribution function is required to determine the evolution of a constituent particle. The transition probability is time dependent and not stationary.

In this paper we introduce a stationary Markov process that enables us to treat all particles independently, by extending the phase space by a phase factor $U(1)$ on an odd lattice. The process has branching and vanishing points and a finite time interval can be maintained between the branchings.

2. The stationary quantum Markov process for the Wigner function

2.1. Wigner function on a lattice

Let N be an odd number. We define $N \times N$ phase and shift matrices Q, P as

$$Q = (Q_{nm}) \equiv (\omega^n \bar{\delta}_{n,\overline{m}}) \begin{pmatrix} \omega^{\frac{N}{2}-\frac{1}{2}} & & & & & & \\ & \cdot & & & & & \\ & & \omega & & & & \\ & & & 1 & & & \\ & & & & \omega^{-1} & & \\ & & & & & \cdot & \\ & & & & & & \omega^{-\frac{N}{2}+\frac{1}{2}} \end{pmatrix}, \tag{4}$$

$$P = (P_{nm}) \equiv (\bar{\delta}_{n+1,m}) = \begin{pmatrix} 0 & & & & & & 1 \\ 1 & \cdot & & & & & \\ & \cdot & \cdot & & & & \\ & & 1 & 0 & & & \\ & & & 1 & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & & 1 & 0 \end{pmatrix}, \tag{5}$$

where $m, n \in Z_N$ and are labeled from $(N-1)/2$ to $-(N-1)/2$ with step-1, $\bar{\delta}$ is the Kronecker delta function modulo N, ω is a primitive Nth root of unity, e.g. $\exp(2\pi i/N)$. The matrices Q, P correspond to the phase and shift operators

$$e^{ia\hat{q}}, \qquad e^{ia\hat{p}}, \tag{6}$$

respectively, in continuous spacetime, where a is a lattice constant. The Weyl matrices $W(m, n)$ and Fano matrices $\Delta(m, n)$ are defined as

$$W(m, n) = \omega^{-2mn} Q^{2n} P^{-2m}, \tag{7}$$

$$\Delta(m, n) = W(m, n)T, \tag{8}$$

where T is a matrix given by

$$T = (T_{nm}) \equiv (\bar{\delta}_{n+m,0}) = \begin{pmatrix} & & & & 1 \\ & & & \cdot & \\ & & \cdot & & \\ & 1 & & & \\ & \cdot & & & \\ \cdot & & & & \\ 1 & & & & \end{pmatrix}. \tag{9}$$

The Weyl matrices $W(m, n)$ can also be written as

$$W(m, n) = \omega^{2mn} P^{-2m} Q^{2n} = P^{-m} Q^{2n} P^{-m} = Q^n P^{-2m} Q^n. \tag{10}$$

The N^2 Weyl matrices and Fano matrices are both complete and orthonormal in the trace norm:

$$\mathrm{Tr}[W^\dagger(m, n)W(m, n)] = N\delta_{m,n},$$
$$\mathrm{Tr}[\Delta^\dagger(m, n)\Delta(m, n)] = N\delta_{m,n}, \tag{11}$$
$$W^\dagger(m, n) = W(-m, -n), \qquad \Delta^\dagger(m, n) = \Delta(m, n),$$

so that arbitrary matrices can be expressed as a linear combination of these matrices. We call the coefficients that appear in the Weyl and Fano expansions the Weyl and Fano coefficients, respectively.

In the continuous case, the Weyl operator is defined as

$$\begin{aligned} W_c(q, p) &= 2e^{-2ipq/\hbar} Q_c^{2p} P_c^{-2q} = 2e^{2ipq/\hbar} P_c^{-2q} Q_c^{2p} \\ &= 2Q_c^p P_c^{-2q} Q_c^p = 2P_c^{-q} Q_c^{2p} P_c^{-q} \end{aligned} \tag{12}$$

where

$$Q_c = e^{i\hat{q}}, \qquad P_c = e^{i\hat{p}}. \tag{13}$$

The Wigner function is written as

$$\rho_c(q, p) = \frac{1}{2\pi\hbar} \mathrm{Tr}[\rho W_c(q, p) T_c] \tag{14}$$

where the operator T_c is given by

$$T_c \equiv \int_{-\infty}^{\infty} dq \, |-q\rangle\langle q|. \tag{15}$$

The Wigner function on a lattice is defined in an analogous way to the continuous case:

$$\rho(m, n) = \frac{1}{N} \mathrm{Tr}[\rho W(m, n) T] = \frac{1}{N} \mathrm{Tr}[\rho \Delta(m, n)]. \tag{16}$$

It is easily seen that this definition satisfies the marginality condition on the lattice.

2.2. Quantum Liouville equation for the Wigner function

Let $\tilde{H}(m, n)$ be the Weyl coefficients of the Hamiltonian,

$$H = \sum_{m,n} W(m, n)\tilde{H}(m, n), \tag{17}$$

and $\rho(m, n)$ be the Fano coefficients of the density matrix, i.e. the Winger function,

$$\rho = \sum_{m,n} \Delta(m, n)\rho(m, n). \tag{18}$$

The quantum Liouville equation

$$i\hbar \frac{d}{dt}\rho = [H, \rho] \tag{19}$$

can be written using these coefficients as

$$\frac{d}{dt}\rho(m, n) = \frac{1}{i\hbar} \sum_{m',n'} \left\{ \tilde{H}(m - m', n - n')\omega^{-2(mn'-nm')} \right.$$
$$\left. -\tilde{H}^*(m - m', n - n')\omega^{2(mn'-nm')} \right\}\rho(m', n'). \tag{20}$$

We modify the equation as follows for later convenience. We perform a polar decomposition of the Weyl coefficients of the Hamiltonian:

$$\tilde{H}(m, n) = \tilde{h}(m, n)\omega^{\theta(m,n)}, \tag{21}$$

where $\theta(0, 0)$ is 0 from the Hermiticity of the Hamiltonian. The time evolution equation for the Wigner function (20) can be rewritten by this decomposition as follows:

$$\frac{d}{dt}\rho(m, n) = \frac{1}{\hbar} \sum_{m',n'} \tilde{h}(m - m', n - n')$$
$$\times \left\{ \omega^{-2(mn'-nm')+\theta(m-m',n-n')-N/4} + \omega^{2(mn'-nm')-\theta(m-m',n-n')+N/4} \right\}\rho(m', n'). \tag{22}$$

The term on the right-hand side equals zero when $m = m'$, $n = n'$, so that it can be omitted in the sum.

2.3. Generator of the Markov process and time evolution equation for φ

We extend the phase space $Z_N \times Z_N$ with the phase factor $U(1)$ and consider a non negative function $\varphi(\alpha, m, n)$ on $U(1) \times Z_N \times Z_N$ that has the following relation with the Wigner function:

$$\rho(m, n) = \int d\alpha \, e^{i\alpha}\varphi(\alpha, m, n). \tag{23}$$

Since the Wigner function is bounded in modulus by $1/N$ on a lattice

$$-\frac{1}{N} \leqslant \rho(m, n) \leqslant \frac{1}{N},$$

the function $\varphi(\alpha, m, n)$ exists, although it is not uniquely determined. We can take

$$\varphi(\alpha, m, n) = \varphi(-\alpha, m, n) \tag{24}$$

without loss of generality as $\rho(m, n)$ is real.

We consider the following time evolution equation for $\varphi(\alpha, m, n)$ within a short time period Δt:

$$\varphi(\alpha, m, n; t + \Delta t) = \int d\alpha' \sum_{m',n'} M_{\Delta t}(\alpha, m, n; \alpha', m', n')\varphi(\alpha', m', n'; t) \quad (25)$$

where $M_{\Delta t}(\alpha, m, n; \alpha', m', n')$ is given by

$$M_{\Delta t}(\alpha, m, n; \alpha', m', n') = (1 - D\Delta t)\delta(\alpha - f_{\Delta t}(\alpha'))\delta_{m,m'}\delta_{n,n'} + \frac{D\Delta t}{2|\tilde{h}|}\tilde{h}(m - m', n - n')$$

$$\times \{\delta(\alpha - \alpha' - 2\pi(-2(mn' - nm') + \theta(m - m', n - n') - N/4)/N)$$

$$+ \delta(\alpha - \alpha' - 2\pi(+2(mn' - nm') - \theta(m - m', n - n') + N/4)/N)\}, \quad (26)$$

with

$$|\tilde{h}| \equiv \sum_{m,n} \tilde{h}(m, n). \quad (27)$$

We assume the function $f_{\Delta t}(\alpha)$ in this equation satisfies

$$(1 - D\Delta t)\cos f_{\Delta t}(\alpha) = \cos \alpha, \quad (28)$$

$$\lim_{\Delta t \to 0} f_{\Delta t}(\alpha) = \alpha. \quad (29)$$

The definition of $M_{\Delta t}(\alpha, m, n; \alpha', m', n')$ shows that particles with ratio $D\Delta t$ move to other phase points with probability $\tilde{h}(m - m', n - n')/|h|$ within Δt. On the other hand, other particles stay at the same phase points and change their phase from α to $f_{\Delta t}(\alpha)$ deterministically. This means that $M_{\Delta t}$ represents the time evolution of a Markov process for $\varphi(\alpha, m, n)$.

2.4. Time evolution of the Wigner function

We consider the time evolution of the Wigner function $\rho(m, n)$ when $\varphi(\alpha, m, n)$ evolves following equation (25):

$$\rho(m, n; t + \Delta t) = \int d\alpha \int d\alpha' \sum_{m',n'} e^{i\alpha} M_{\Delta t}(\alpha, m, n; \alpha', m', n')\varphi(\alpha', m', n'; t)$$

$$= \rho(m, n; t) + \frac{D\Delta t}{2|\tilde{h}|} \int d\alpha' \sum_{m',n'} \tilde{h}(m - m', n - n')$$

$$\times (e^{i\alpha'}\omega^{-2(mn'-nm')+\theta(m-m',n-n')-N/4}$$

$$+ e^{i\alpha'}\omega^{+2(mn'-nm')-\theta(m-m',n-n')+N/4})\varphi(\alpha', m', n'; t). \quad (30)$$

We have

$$\frac{1}{\Delta t}\{\rho(m, n; t + \Delta t) - \rho(m, n; t)\} = \frac{D}{2|\tilde{h}|} \sum_{m',n'} \tilde{h}(m - m', n - n')$$

$$\times (\omega^{-2(mn'-nm')+\theta(m-m',n-n')-N/4} + \omega^{+2(mn'-nm')-\theta(m-m',n-n')+N/4})$$

$$\times \int d\alpha' \, e^{i\alpha'}\varphi(\alpha', m', n'; t) \quad (31)$$

and in the $\Delta t \to 0$ limit

$$\frac{d}{dt}\rho(m, n) = \frac{D}{2|\tilde{h}|i} \sum_{m',n'} \tilde{h}(m - m', n - n')$$

$$\times (\omega^{-2(mn'-nm')+\theta(m-m',n-n')-N/4} + \omega^{+2(mn'-nm')-\theta(m-m',n-n')+N/4})\rho(m', n'). \quad (32)$$

This shows that if $\varphi(\alpha, m, n)$ evolves following the time evolution equation (25) and we take

$$D/2|\tilde{h}| = 1/\hbar, \tag{33}$$

the function $\rho(m, n)$ obtained from the distribution function $\varphi(\alpha, m, n)$ satisfies the quantum Liouville equation for the Wigner function.

2.5. Generator of the Markov process for φ

The generator of a Markov process $A(\alpha, m, n; \alpha', m', n')$ is generally defined by

$$A(\alpha, m, n; \alpha', m', n') \equiv \lim_{\Delta t \to 0} \frac{M_{\Delta t} - I}{\Delta t}. \tag{34}$$

The generator corresponding to $M_{\Delta t}(\alpha, m, n; \alpha', m', n')$ is given by

$$
\begin{aligned}
A(\alpha, m, n; \alpha', m', n') = {} & \frac{D}{\tan \alpha'} \delta'(\alpha - \alpha') \delta_{m,m'} \delta_{n,n'} \\
& - D\delta(\alpha - \alpha') \delta_{m,m'} \delta_{n,n'} + \frac{D}{2|\tilde{h}|} \tilde{h}(m - m', n - n') \\
& \times \{\delta(\alpha - \alpha' - 2\pi(-2(mn' - nm') + \theta(m - m', n - n') - N/4)/N) \\
& + \delta(\alpha - \alpha' - 2\pi(+2(mn' - nm') - \theta(m - m', n - n') + N/4)/N)\}.
\end{aligned}
\tag{35}
$$

This generator is singular at $\alpha = 0, \pi$. These singularities come from the fact that when we set

$$f_{\Delta t}(\alpha) = \alpha + \Delta \alpha \tag{36}$$

in equations (28) and equation (29), we have

$$\Delta t / \Delta \alpha = -\frac{1}{D} \tan \alpha = 0 \tag{37}$$

and Δt becomes zero at these points. This means we cannot take the time evolution for any choice of $\Delta \alpha$ at these points. To avoid this difficulty, we have to choose the initial $\varphi(\alpha, m, n)$ to be zero at $\alpha = 0, \pi$ and perform branching without changing the contribution to the Wigner function when a particle reaches these points. For example, a particle which reaches $\alpha = 0$ can be replaced by two particles at $\alpha = +\pi/3$ and $\alpha = -\pi/3$.

In the limit $\Delta t \to 0$, the time evolution equation for $\varphi(\alpha, m, n)$ satisfies

$$
\begin{aligned}
\dot{\varphi}(\alpha, m, n) = {} & \frac{\partial}{\partial \alpha} \frac{D}{\tan \alpha} \varphi(\alpha, m, n) - D\varphi(\alpha, m, n) \\
& + \int d\alpha' \sum_{m',n'} \frac{D}{2|\tilde{h}|} \tilde{h}(m - m', n - n') \\
& \times \{\delta(\alpha - \alpha' - 2\pi(-2(mn' - nm') + \theta(m - m', n - n') - N/4)/N) \\
& + \delta(\alpha - \alpha' - 2\pi(+2(mn' - nm') - \theta(m - m', n - n') + N/4)/N)\} \\
& \times \varphi(\alpha', m', n').
\end{aligned}
\tag{38}
$$

The quantum Liouville equation for the Wigner function $\rho(m, n)$ can also be derived directly from this equation. We observe that equation (38) reduces to equation (22) if the part

$$\frac{\partial}{\partial \alpha} \frac{D}{\tan \alpha} \varphi(\alpha, m, n) - D\varphi(\alpha, m, n) \tag{39}$$

vanishes after the integration with respect to α multiplying $e^{i\alpha}$. Paying attention to $\varphi(\alpha) = \varphi(-\alpha)$, the integration

$$\int_0^{2\pi} e^{i\alpha} \frac{\partial}{\partial \alpha} \frac{1}{\tan \alpha} \varphi(\alpha) \, d\alpha$$

becomes

$$2 \int_0^\pi \cos \alpha \frac{\partial}{\partial \alpha} \frac{1}{\tan \alpha} \varphi(\alpha) \, d\alpha = 2[\sin \alpha \varphi(\alpha)]_0^\pi - 2 \int_0^\pi \frac{\partial \cos \alpha}{\partial \alpha} \frac{1}{\tan \alpha} \varphi(\alpha) \, d\alpha$$

$$= 2 \int_0^\pi \cos \alpha \varphi(\alpha) \, d\alpha = \int_0^{2\pi} e^{i\alpha} \varphi(\alpha) \, d\alpha \qquad (40)$$

and equation (22) holds.

3. Construction procedure

In the following, we summarize the procedure to determine the time evolution of the Wigner function from the Markov process of particles with distribution function $\varphi(\alpha, m, n)$.

3.1. Preparation

(1) Prepare an additional space $U(1)$ on each phase-space point (m, n).
(2) Calculate the Wigner function from the initial wavefunction.
(3) Expand the Wigner function into the distribution function and make an appropriate choice of $\varphi(\alpha, m, n)$.
(4) Perform the Weyl expansion of the Hamiltonian and calculate the Weyl coefficients $\tilde{H}(m, n)$.
(5) Perform the polar decomposition of $\tilde{H}(m, n)$ and calculate $\theta(m, n)$.

3.2. Rules of the process

(1) Particles of ratio $D\Delta t$ move to other phase points within Δt. Others stay at the same phase point.
(2) The probability $P(m, n; m', n')$ for moving from (m', n') to (m, n) is determined by

$$P(m, n; m', n') = \tilde{h}(m - m', n - n')/|\tilde{h}|, \qquad (41)$$

with equation (27).
(3) The phase change is deterministic for particles without jumps to other phase points. The phase is rotated from α to $f_{\Delta t}(\alpha)$.

3.3. Simulation

(1) Prepare an ensemble of particles with the initial distribution function.
(2) Make transitions for each particle independently in the extended phase space following the probability rule above within a period of time.
(3) For a jump to another phase point, add the phases $\omega^{-2(mn'-nm')+\theta(m-m',n-n')-N/4}$ and $\omega^{2(mn'-nm')-\theta(m-m',n-n')+N/4}$ with probability $1/2$.
(4) Pull back the particle when it reaches the stopping point $\alpha = 0, \pi$ with branching.

(5) After performing the process for each particle, integrate the phase part multiplying $e^{i\alpha}$ to the density distribution function $\varphi(\alpha, m, n)$. The resultant function is the Wigner function $\rho(m, n)$ under equation (33).

The ensemble must be remade if the number of particles increases.

We show an example for $N = 3$ just as a clue to applications. When the Hamiltonian and initial wavefunction are given by

$$H = S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad |\psi\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix},$$

with

$$\langle S_x \rangle = 1, \qquad \langle S_y \rangle = \langle S_z \rangle = 0,$$

where

$$S_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_y = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

the Wigner function and an extended distribution function are given by

$$\rho(m, n) = \frac{1 - \sqrt{2}}{12}, \qquad \varphi(\alpha, m, n) = \frac{\sqrt{2} - 1}{12}\left(\delta\left(\alpha - \frac{2\pi}{3}\right) + \delta\left(\alpha + \frac{2\pi}{3}\right)\right),$$

for $(m, n) = (1, 1), (1, -1), (-1, 1), (-1, -1)$,

$$\rho(m, n) = \frac{1 + 2\sqrt{2}}{12}, \qquad \varphi(\alpha, m, n) = \frac{1}{12}\left(\delta\left(\alpha - \frac{\pi}{3}\right) + \delta\left(\alpha + \frac{\pi}{3}\right)\right),$$

for $(m, n) = (0, 1), (0, -1)$,

$$\rho(m, n) = 1/12, \qquad \varphi(\alpha, m, n) = \frac{1 + 2\sqrt{2}}{12}\left(\delta\left(\alpha - \frac{\pi}{3}\right) + \delta\left(\alpha + \frac{\pi}{3}\right)\right),$$

for $(m, n) = (1, 0), (-1, 0)$,

$$\rho(m, n) = 1/3, \qquad \varphi(\alpha, m, n) = \frac{1}{\sqrt{3}}\left(\delta\left(\alpha - \frac{\pi}{3}\right) + \delta\left(\alpha + \frac{\pi}{3}\right)\right),$$

for $(m, n) = (0, 0)$. The Weyl coefficients have only two non-zero values, i.e.,

$$\tilde{H}(0, 1) = i/\sqrt{3}, \qquad \tilde{H}(0, -1) = -i/\sqrt{3},$$

hence the stochastic process is simply determined from

$$P(0, 1) = P(0, -1) = 1/2, \qquad \theta(0, 1) = \theta(0, -1) = 3/2.$$

4. Summary

If we consider a lattice constituting of N odd lattice points as a cyclic group of order N, the corresponding phase space is $Z_N \times Z_N$. By extending the phase space by a phase factor $U(1)$, we can construct a stationary Markov process in the space $U(1) \times Z_N \times Z_N$. This process has the property that multiplying the phase factor to the density distribution function of the process and integrating the phase part gives a function which satisfies the quantum Liouville equation for the Wigner function.

There exist branching points at the phase $0, \pi$, but we can maintain a finite time interval before the branching by making an appropriate choice of the initial condition and the branching

method. There is no contribution from the distribution at the phase $\pm\pi/2$. The process does not need to be continued for particles reaching these points as the stochastic rule is symmetric around these points and they give no contribution at future times. In this sense, these points can be treated as vanishing points. The estimation of branching and vanishing ratio is a problem for future studies.

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